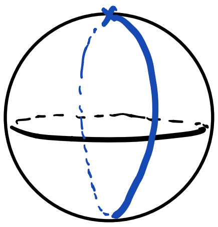
§ Spaces Forms

There are 3 "model geometries" in 2D:

($K \equiv 1$)
Spherical

(S^2, g_{round})

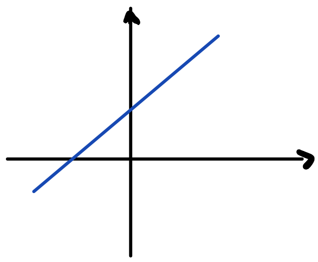


$\downarrow$

$\mathbb{RP}^2$

($K \equiv 0$)
Euclidean

$(\mathbb{R}^2, g_{flat})$

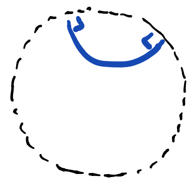


$\downarrow$

T^2, $\mathbb{R} \times S^1$

($K \equiv -1$)
Hyperbolic

$(\mathbb{H}^2, g_{hyp})$



Poincaré disk model

$\downarrow$

Σ_g $(g \geq 2)$

Q: Can we classify the "model geometries" in higher dimensions?

ie constant curvature spaces

Cartan Theorem: S^n, $\mathbb{R}^n$, $\mathbb{H}^n$ are the only simply connected, complete Riem. n-manifolds with constant sectional curvatures.

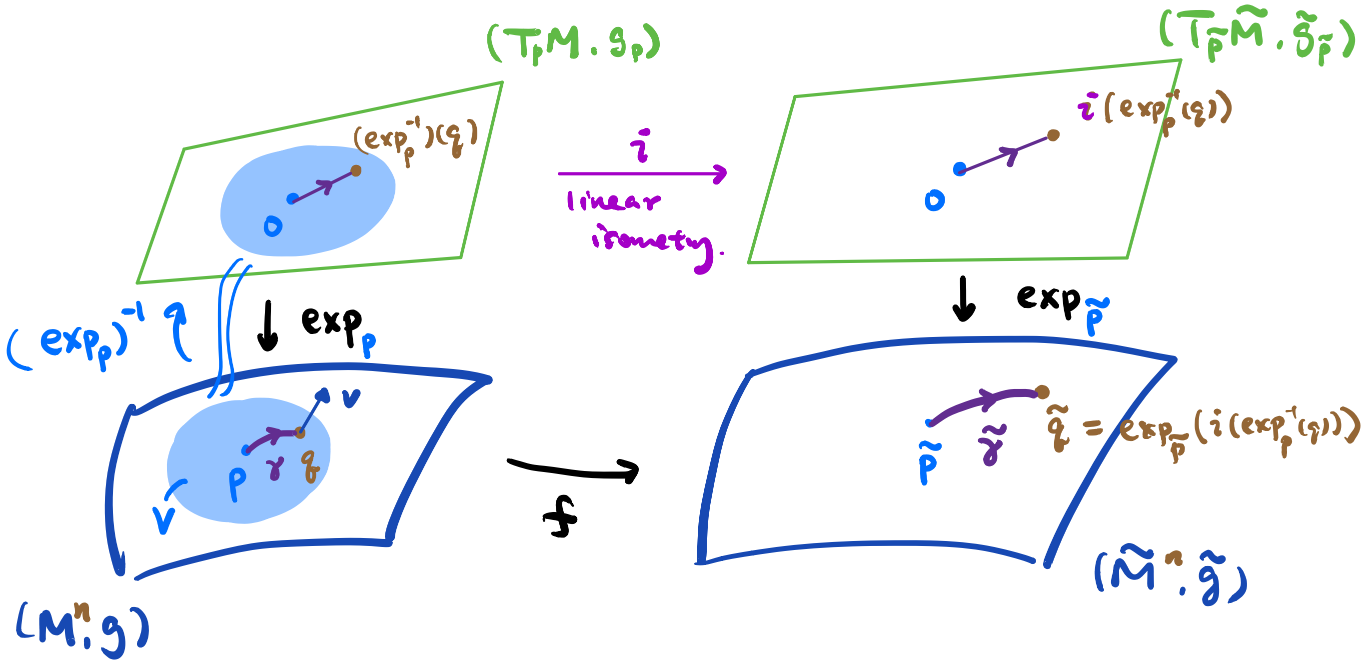
The key idea to the proof is a lemma due to Cartan-Ambrose which says that the Riem. curvature tensor R determines the Riem. metric g locally.

$$g \underset{\text{"}\int\text{"}}{\overset{\text{"}d\text{"}}{\rightleftarrows}} R$$

Notation for Cartan-Ambrose Lemma:

(M^n, g), $(\tilde{M}^n, \tilde{g})$ — complete Riem. mfd of same dim $= n$

Fix $p \in M$, $\tilde{p} \in \tilde{M}$ and a linear isometry $i : T_pM \to T_{\tilde{p}}\tilde{M}$



Let $V \subseteq M$ be a nbd of p in M st the geodesic normal coordinate system (centered at p) is well-defined in V

Define: $f : V \to \tilde{M}$ by $f(q) := \exp_{\tilde{p}} \circ i \circ \exp_p^{-1}(q)$

Let $\gamma : [0, t] \to M$ be geodesic from p to q

$\tilde{\gamma} : [0, t] \to M$ be geodesic from $\tilde{p}$ to $\tilde{q}$

parallel transports along γ, $\tilde{\gamma}$

Define: $\phi_t : T_qM \to T_{\tilde{q}}\tilde{M}$ by $\phi_t(v) := \tilde{P}_t \circ i \circ P_t^{-1}(v)$

Cartan-Ambrose Lemma: Under the notations above:

If $\forall q \in V$, $\forall x, y, u, v \in T_q M$.

$$R(x,y,u,v) = \tilde{R}(\phi_t(x), \phi_t(y), \phi_t(u), \phi_t(v)).$$

THEN, $f : V \to f(V)$ is a local isometry.

"Sketch of Proof": Fix $q \in V$, and let $\gamma : [0, \ell] \to M$ p.b.a.l..

Fix $v \in T_q M$. By the choice of V, q is NOT conjugate to p

$\Rightarrow \exists$ Jacobi field $V(t)$ along $\gamma(t)$ st $V(0) = 0$, $V(\ell) = v$

Choose a parallel O.N.B. $\{e_1, \ldots e_n\}$ along γ st $e_n = \gamma'$

write: $$V(t) = \sum_{j=1}^{n} \alpha_j(t)\, e_j(t)$$

Jacobi eq$^{\underline{n}}$ $\Rightarrow$ $$\alpha_j'' + \sum_{i=1}^{n} R(e_n, e_i, e_n, e_j)\, \alpha_i = 0 \quad \text{for } j = 1, \ldots, n$$

Define: $\tilde{V}(t) := \phi_t(V(t)) \quad \forall t \in [0, \ell]$

$\tilde{e}_j(t) := \phi_t(e_j(t))$ O.N.B. parallel along $\tilde{\gamma}$

Note: $$\tilde{V}(t) = \sum_{j=1}^{n} \alpha_j(t)\, \tilde{e}_j(t)$$

By hypothesis, $R(e_n, e_i, e_n, e_j) = \tilde{R}(\tilde{e}_n, \tilde{e}_i, \tilde{e}_n, \tilde{e}_j)$.

$$\Rightarrow \alpha_j'' + \sum_{i=1}^{n} \tilde{R}(\tilde{e}_n, \tilde{e}_i, \tilde{e}_n, \tilde{e}_j)\, \alpha_i = 0 \quad \text{for } j = 1, \ldots, n$$

So, $\tilde{V}(t)$ is a Jacobi field along $\tilde{\gamma}$ with $\tilde{V}(0) = 0$.

Since P_t, $\tilde{P}_t$ are isometries, we have $\|\tilde{V}(\ell)\| = \|V(\ell)\|$

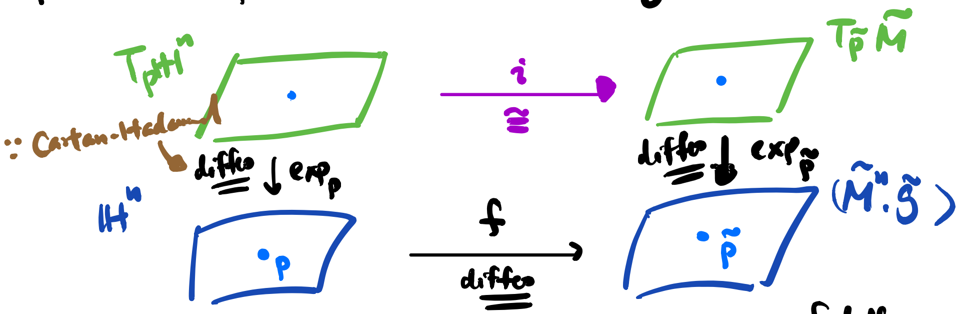
Finally, one checks (Ex.) that $\tilde{V}(\ell) = df_q(v)$, i.e. df_q is isometry.

$\square$

Cartan's Thm: Let (M^n, g) be a complete Riem. manifold. Suppose (M^n, g) has constant sectional curvature $K_0 \in \{1, 0, -1\}$. THEN, the universal cover $(\tilde{M}^n, \tilde{g})$ is isometric to either S^n ($K_0 = 1$), $\mathbb{R}^n$ ($K_0 = 0$) or $\mathbb{H}^n$ ($K_0 = -1$).

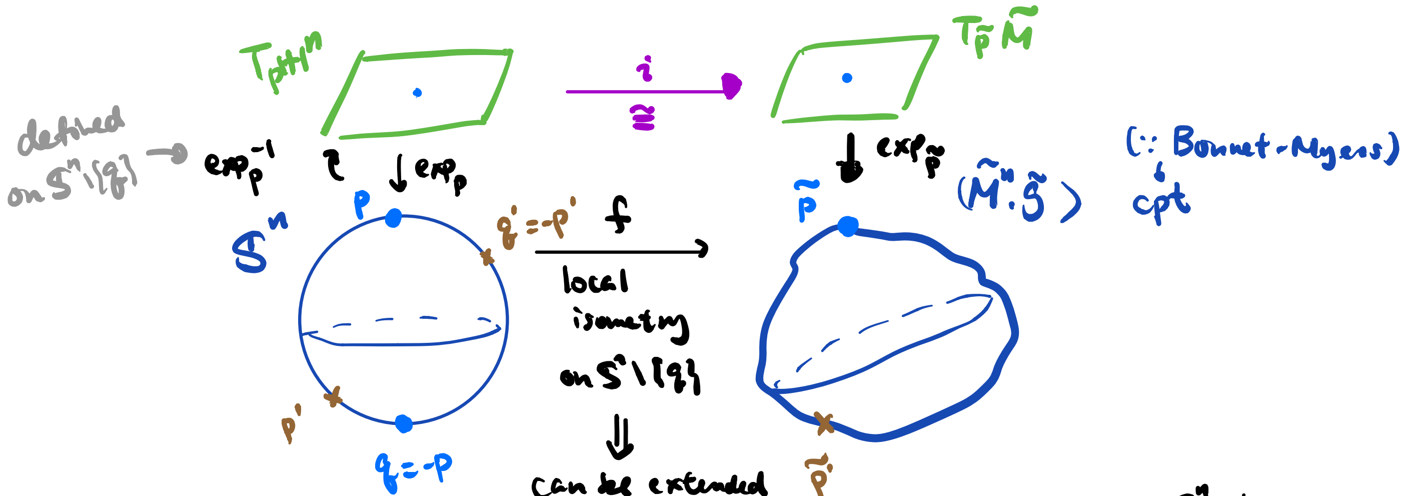
"**Proof**": **Case 1**: $K_0 = -1$ or $K_0 = 0$

Fix $p \in \mathbb{H}^n$, $\tilde{p} \in \tilde{M}$ and isometry $i : T_p\mathbb{H}^n \to T_{\tilde{p}}\tilde{M}$



Cartan-Ambrose Lemma $\Rightarrow$ f local isometry $\xRightarrow{f \text{ diffeo}}$ f global isometry

Case 2: $K_0 = 1$ Fix $p \in \mathbb{H}^n$, $\tilde{p} \in \tilde{M}$ and isometry $i : T_p\mathbb{H}^n \to T_{\tilde{p}}\tilde{M}$



can be extended across q
remains local isometry $\Rightarrow$ covering map $\xRightarrow{\because S^n \text{ simply connected}}$ f diffeo.

$\square$